SYMPLECTIC ACTIONS OF NON-HAMILTONIAN TYPE

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In memory of Professor Johannes (Hans) J. Duistermaat (1942–2010)

ABSTRACT. Hamiltonian symplectic actions of tori on compact symplectic manifolds have been extensively studied in the past thirty years, and a number of classifications have been achieved, for instance in the case that the acting torus is n-dimensional and the symplectic manifold is $2n$-dimensional. In this case the n-dimensional orbits are Lagrangian, so it is natural to wonder whether there are interesting classes of symplectic actions with Lagrangian orbits, and that are not Hamiltonian. It turns out that there are many such classes which contain for example the Kodaira variety, and which can be classified in terms of symplectic invariants. The paper reviews several classifications, which include symplectic actions having a Lagrangian orbit or a symplectic orbit of maximal dimension. We make an emphasis on the construction of the symplectic invariants, and their computation in examples.

1. INTRODUCTION

The study of Hamiltonian symplectic actions in the past thirty years has been a driving force in equivariant symplectic geometry, and there is an extensive theory, which includes several classification results. It is natural to wonder whether there are interesting classes of actions that are symplectic, but not Hamiltonian, and if so, whether they may be viewed as part of a larger cohesive framework. From a geometric view point, being only symplectic is a natural assumption, and while Hamiltonian actions of maximal dimension appear as symmetries in many integrable systems in mechanics, non-Hamiltonian actions also occur in physics, see eg. Novikov [28], as well as in complex algebraic geometry, and in topology.

In this paper we review the classical results about symplectic Hamiltonian actions by Atiyah, Guillemin-Sternberg, and Delzant [1, 18, 5], and then present a number of recent results (since around 2002) on the classification of certain classes of non-Hamiltonian symplectic torus actions, and which are contained in the work Benoist, Duistermaat, and the author [2, 3, 7, 33]. We outline connections of these works with algebraic geometry (in particular Kodaira's classification of complex analytic surfaces [22]), with topology (in particular the work of Benson-Gordon [4] on torus bundles over tori and nilpotent Lie groups), with the work of Guillemin-Sternberg on multiplicity-free spaces [19], and also with orbifold theory (for instance Thurston's classification of compact 2-dimensional orbifolds).

1

Throughout the paper we emphasize the role that the Hamiltonian theory of Atiyah et al. plays in the non-Hamiltonian theory; essentially, when a non-Hamiltonian torus action has a subtorus which acts in a Hamiltonian fashion, the Hamiltonian theory can be applied to symplectically characterize the action of such subtori in terms of a combinatorial invariant (a polytope).

The proofs of the results by Atiyah et al. are covered from a number of view points in the literature (see for instance Guillemin's book [15]), and we will not attempt to discuss them further. Instead, we will describe in a way accessible to non experts the recent symplectic classifications in [2, 3, 7, 33], making an emphasis in the construction of the symplectic invariants that appear in the classification theorems. We also carry out the computation of the invariants in examples such as the Kodaira variety [22] (also known as the Kodaira-Thurston manifold [39]).

A *symplectic manifold* is a smooth manifold M equipped with a smooth non-degenerate closed 2-form ω, called a *symplectic form*. Equivariant symplectic geometry is concerned with the study of smooth actions of Lie groups G on symplectic manifolds M, by means of diffeomorphisms $\varphi \in \mathrm{Diff}(M)$ which pull-back the symplectic form ω to itself $\varphi^*\omega = \omega$ (these are called *symplectomorphisms*). Actions satisfying this natural condition are called *symplectic*. In this paper we treat the case when G is a compact, connected, abelian Lie group, that is, a torus: $T \simeq (S^1)^k$, $k \geq 1$. Let $\mathfrak{t}$ be the Lie algebra of T, and let $\mathfrak{t}^*$ be its dual Lie algebra. A fundamental subclass of symplectic actions admit what is called al *momentum map*, which is a $\mathfrak{t}^*$-valued smooth function on M which encodes information about M itself, the symplectic form, and the action; such symplectic actions are called *Hamiltonian*.

The category of Hamiltonian actions, while large, does not include some simple examples of symplectic actions, for instance free symplectic actions on compact manifolds, because Hamiltonian actions on compact manifolds always have fixed points. One striking (and basic) open question is: are there non-Hamiltonian symplectic S^1-actions on compact connected manifolds with non-empty *and* discrete fixed point set? In recent years there has been a flurry of activity related to this question, see for instance Godinho [13, 14], Jang [20, 21], Li-Liu [25], Pelayo-Tolman [36], Sabatini [38], and Tolman-Weitsman [40]; the answer is unknown.

In the present paper we will not discuss this question because the dimension of the torus is one, and instead we focus on symplectic actions of tori of dimension k on manifolds of dimension $2n$ where $k \geq n$. When in addition to being symplectic the action is Hamiltonian, then necessarily $n = k$, but there are many non-Hamiltonian symplectic actions when $n = k$, and also when $n \geq k + 1$.

We will concentrate on smplectic non-Hamiltonian actions in three distinct cases: when the manifold is four-dimensional, when there is an orbit of maximal dimension which is symplectic, and when there is a coisotropic orbit (these last two cases are disjoint from each other, and the four dimensional

case overlaps with both of them). The moduli space of symplectic actions satisfying one of these conditions is large, and includes as a particular case Hamiltonian actions of maximal dimension (see [34] for the description of the moduli space of Hamiltonian actions of maximal dimension on 4-manifolds).

Some of the techniques to study Hamiltonian torus actions (see for instance the books by Guillemin [15], Guillemin-Sjamaar [17], and Ortega-Ratiu [30]) are useful in the study of non-Hamiltonian symplectic torus actions (since many non-Hamiltonian actions exhibit proper subgroups which act Hamiltonianly). In the study of Hamiltonian actions, one tool that is often used is Morse theory for the (components of the) momentum map of the action. Since there is no momentum map in the classical sense for a general symplectic action, Morse theory does not appear as a natural tool in the non-Hamiltonian case. There is an analogue, however, *circle valued-Morse theory* (since any symplectic circle action admits a circle-valued momentum map, see McDuff [27] and [35], which is also Morse in a sense) but it is less immediately useful in our setting; for instance a more complicated form of the Morse inequalities holds (see Pajitnov [31, Chapter 11, Proposition 2.4] and Farber [11, Theorem 2.4]), and the theory appears more difficult to apply, at least in the context of non-Hamiltonian symplectic actions; see [35, Remark 6] for further discussion in this direction. This could be one reason that non-Hamiltonian symplectic actions have been studied less in the literature than their Hamiltonian counterparts. However, as this paper shows, there are rich classes of non-Hamiltonian symplectic actions, which include examples of interest not only in symplectic geometry, but also in algebraic geometry (eg. the Kodaira variety), differential geometry (eg. multiplicty free spaces), and topology (eg. nilmanifolds over nilpotent Lie groups, orbifold bundles). For instance, a compact symplectic manifold endowed with a symplectic action with Lagrangian orbits (eg. the Kodaira variety) is characterized in terms of four symplectic invariants (Theorem 3.9); if the action is also Hamiltonian, only one invariant remains (the polytope Δ).

Structure of the paper. In Section 2 we review some of the foundational results on symplectic Hamiltonian torus actions on compact manifolds; the theory is well documented in the literature, so we limit ourselves to the core aspects. In Section 3 we discuss the classification of symplectic actions when there exists a coisotropic orbit. In Section 4, we explain the classification when there exists a symplectic orbit of the same dimension as the acting torus. In Section 5 we state the classification of symplectic actions of 2-tori on 4-manifolds. In Section 6 we outline the proof of a key result in Section 3. Because this paper is introductory we avoid technical statements; references are given throughout for those interested in further details.

2. Symplectic torus actions of Hamiltonian type

This section treats the case when the action, in addition to being symplectic, is Hamiltonian. Let (M, ω) be compact, connected, $2n$-dimensional

symplectic manifold. Let T be a torus. Suppose that T acts effectively and symplectically on M. Recall the meaning of these notions: a differential 2-form ω on M is *symplectic* if it is closed, i.e. $\mathrm{d}\omega = 0$, as well as non-degenerate. The action $T \times M \to M$ is *effective* if every element in the torus T moves at least one point in M, or equivalently $\bigcap_{x \in M} T_x = \{e\}$, where $T_x := \{t \in T \mid t \cdot x = x\}$ is the *stabilizer subgroup of the T-action at x*. The action is *free* if $T_x = \{e\}$ for every $x \in M$. The action $T \times M \to M$ is *symplectic* if T acts by symplectomorphisms, i.e. diffeomorphisms $\varphi \colon M \to M$ such that $\varphi^*\omega = \omega$.

A type of symplectic actions are *Hamiltonian* actions. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{t}^*$ its dual. A symplectic action $T \times M \to M$ is *Hamiltonian* if there is a smooth map $\mu \colon M \to \mathfrak{t}^*$ such that Hamilton's equation

$$(2.1) \qquad\qquad -\mathrm{d}\langle \mu,\, X \rangle = \mathrm{i}_{X_M}\omega := \omega(X_M, \cdot), \quad \forall X \in \mathfrak{t},$$

holds, where X_M is the vector field infinitesimal action of X on M, and the right hand-side of equation (2.1) is the one-form obtained by pairing of the symplectic form ω with X_M, while the left hand side is the differential of the real valued function $\langle \mu(\cdot),\, X \rangle$ obtained by evaluating elements of $\mathfrak{t}^*$ on $\mathfrak{t}$. The Lie algebra $\mathfrak{t}_x$ of T_x is the kernel of the linear mapping $X \mapsto X_M(x)$ from $\mathfrak{t}$ to $\mathrm{T}_x M$. In the upcoming sections we will use use the notation $\mathfrak{t}_M(x) := \mathrm{T}_x(T \cdot x)$, where $T \cdot x := \{t \cdot x \mid t \in T\}$ is the *T-orbit* that goes through the point x.

It follows from equation (2.1) that Hamiltonian T-actions on compact connected manifolds have fixed points. The Atiyah-Guillemin-Sternberg Convexity Theorem (1982, [1, 18]) says that $\mu(M)$ is the convex hull of the image under μ of the fixed point set of the T-action. The polytope $\mu(M)$ is called the *momentum polytope of M*. One precedent of this result appears in Kostant's article [23]. Later Delzant (1988, [5]) proved that if the action if effective and $2\dim T = 2n = \dim M$ (in which case the triple (M, ω, T) is called a *Delzant manifold* or a *symplectic-toric manifold*), then $\mu(M)$ is a Delzant polytope (i.e. a simple, edge-rational, and smooth polytope; or in other words, there are precisely n-codimension 1 faces meeting at each vertex and their normal vectors span a $\mathbb{Z}$-basis of the integral lattice $T_{\mathbb{Z}} := \ker \exp \colon \mathfrak{t} \to T$). Moreover, Delzant proved that $\mu(M)$ classifies $(M,\, \omega,\, T)$ in the sense of uniqueness (two Delzant manifolds are T-equivariantly symplectomorphic if and only if they have the same associated momentum polytope), and existence (for each abstract Delzant polytope Δ in $\mathfrak{t}^*$ there exists a Delzant manifold with momentum polytope Δ). The $\dim T$-dimensional orbits of a symplectic-toric manifold are Lagrangian submanifolds, that is, the symplectic form vanishes along them.

The simplest example of a Hamiltonian torus action is $(S^2,\, \omega = \mathrm{d}\theta \wedge \mathrm{d}h)$ equipped with the rotational circle action $\mathbb{R}/\mathbb{Z}$ about the vertical axis of S^2. This action has momentum map $\mu \colon S^2 \to \mathbb{R}$ equal to the height function $\mu(\theta,\, h) = h$, and in this case the momentum polytope is the interval $\Delta = [-1,\, 1]$. Another example of a Hamiltonian torus action is the n-dimensional

complex projective space equipped with a λ-multiple, $\lambda > 0$, of the Fubini--Study form $(\mathbb{CP}^n, \lambda \cdot \omega_{\mathrm{FS}})$ and the rotational $\mathbb{T}^n$-action induced from the rotational $\mathbb{T}^n$-action on the $(2n+1)$-dimensional complex plane. This action is Hamiltonian, with momentum map components given by $\mu_k^{\mathbb{CP}^n, \lambda}(z) = \frac{\lambda |z_k|^2}{\sum_{i=0}^{n} |z_i|^2}$ for each $k = 1, \ldots, n$. The associated momentum polytope is $\Delta = $ convex hull $\{0, \lambda \mathrm{e}_1, \ldots, \lambda \mathrm{e}_n\}$, where $\mathrm{e}_1, \ldots, \mathrm{e}_n$ are the canonical basis vectors of $\mathbb{R}^n$. There exists an extensive theory of Hamiltonian actions and related topics, see for instance the books by Guillemin [15], Guillemin-Sjamaar [17], and Ortega-Ratiu [30].

3. SYMPLECTIC TORUS ACTIONS WITH COISOTROPIC ORBITS

This section is based on parts of [7]; see also Benoist [2, 3]. To make the general theory more transparent, we make an emphasis on examples, the connections with algebraic geometry and topology, and the construction of the symplectic invariants. Let (M, ω) be a compact, connected, $2n$--dimensional symplectic manifold. Let T be a torus. Suppose that T acts effectively and symplectically on M.

Coisotropic orbit condition. *We assume throughout this section that there exists a T-orbit which is a coisotropic submanifold.*

3.1. **The meaning of the coisotropic condition.** The *principal orbit type of M*, denoted by M_{reg}, is by definition $M_{\mathrm{reg}} := \{x \in M \mid T_x = \{e\}\}$, or equivalently, M_{reg} is the set of points where the T-action is free. The set M_{reg} is an open dense subset of M, and connected (since T is connected). An orbit of the T-action is *principal* if it lies inside of M_{reg}.

A submanifold $C \subset M$ is *coisotropic* if $(\mathrm{T}_x C)^{\omega_x} \subset \mathrm{T}_x C$ for all $x \in C$, where $(\mathrm{T}_x C)^{\omega_x}$ is the symplectic orthogonal complement to the tangent space to C; recall that if V is a subspace of a symplectic vector space (W, σ), its *symplectic orthogonal complement V^σ* consists of the vectors $w \in W$ such that $\sigma(w, v) = 0$ for all $v \in V$. A submanifold $C \subset M$ is *Lagrangian* if $\omega|_C = 0$ and $\dim C = \frac{\dim M}{2}$; this is a special case of coisotropic submanifold when the inclusion "$\subset$" is an equality "$=$". If C is a coisotropic submanifold of dimension k, then $2n - k = \dim(\mathrm{T}_x C)^{\omega_x} \leq \dim(\mathrm{T}_x C) = k$ shows that $k \geq n$. The submanifold C has the minimal dimension n if and only if $(\mathrm{T}_x C)^{\omega_x} = \mathrm{T}_x C$, if and only if C is a Lagrangian submanifold of M (that is, an isotropic submanifold of M of maximal dimension n).

It is a consequence of the local normal form (i.e. the symplectic tube theorem [7, Section 11]) of Benoist [2, 3] and Ortega-Ratiu [29] that if a symplectic manifold admits a torus action with a coisotropic orbit, then this orbit must be principal. Moreover, the existence of a coisotropic orbit in the principal orbit type implies that all orbits in the principal orbit type are coisotropic. That is, the existence of a single coisotropic orbit is equivalent to all principal orbits being coisotropic.

Remark 3.1. A symplectic torus action for which one can show that it has a Lagrangian orbit falls into the category of actions that we study in this section; this includes Hamiltonian actions of n-tori (see Section 2).

Remark 3.2. Let f, g be in the set of T-invariant smooth functions $C^\infty(M)^T$ and let $x \in M_{\mathrm{reg}}$. Let $\mathcal{H}_f$ be the vector field defined by $i_{\mathcal{H}_f} \omega = -\mathrm{d}f$. Then $\mathcal{H}_f(x)$, $\mathcal{H}_g(x) \in \mathfrak{t}_M(x)^{\omega_x} \cap \mathfrak{t}_M(x)$. It follows that the Poisson brackets $\{f, g\} := \omega(\mathcal{H}_f, \mathcal{H}_g)$ of f and g vanish at x. Since the principal orbit type M_{reg} is dense in M, we have that $\{f, g\} \equiv 0$ for all f, $g \in C^\infty(M)^T$ if the principal orbits are coisotropic. Conversely, if we have that $\{f, g\} \equiv 0$ for all f, $g \in C^\infty(M)^T$, then $\mathfrak{t}_M(x)^{\omega_x} \subset (\mathfrak{t}_M(x)^{\omega_x})^{\omega_x} = \mathfrak{t}_M(x)$ for every $x \in M_{\mathrm{reg}}$, i.e. $T \cdot x$ is coisotropic. Therefore the principal orbits are coisotropic if and only if $\{f, g\} \equiv 0$ for all f, $g \in C^\infty(M)^T$.[1]

Remark 3.3. The coisotropic condition is natural in view of the local models of symplectic torus actions. This observation goes back to Benoist [2].

3.2. **Examples.** Many classical examples of symplectic torus actions have coisotropic principal orbits.

Example 3.4. (Kodaira variety) The first example of a symplectic torus action with coisotropic principal orbits is the Kodaira variety [22] (also known as the Kodaira-Thurston manifold [39]), which is a torus bundle over a torus constructed as follows. Consider the product symplectic manifold $(\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})^2, \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$, where $(x_1, y_1) \in \mathbb{R}^2$ and $(x_2, y_2) \in (\mathbb{R}/\mathbb{Z})^2$. Consider the action of $(j_1, j_2) \in \mathbb{Z}^2$ on $(\mathbb{R}/\mathbb{Z})^2$ by the matrix group consisting of

$$\begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix},$$

where $j_2 \in \mathbb{Z}$. The quotient of this symplectic manifold by the diagonal action of $\mathbb{Z}^2$ gives rise to a compact, connected, symplectic 4-manifold

$$(3.1) \qquad (\mathrm{KT}, \omega) := (\mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2, \ \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2)$$

on which the 2-torus $T := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ acts symplectically and freely, where the first circle acts on the x_1-component, and the second circle acts on the y_2-component (one can check that this action is well defined). We denote the Kodaira variety by KT, and the symplectic form is assumed. Because the T-action is free, all the orbits are principal, and because the orbits are obtained by keeping the x_2-component and the y_1-component fixed, we must have that $\mathrm{d}x_2 = \mathrm{d}y_1 = 0$, and hence the aforementioned form ω vanishes along the orbits, which are therefore Lagrangian submanifolds of KT.

Remark 3.5. The symplectic manifold in formula (3.1) fits in the third case in Kodaira [22, Theorem 19]. Thurston rediscovered it [39], and observed

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[1]In Guillemin and Sternberg [19], a symplectic manifold with a Hamiltonian action of an arbitrary compact Lie group is called a *multiplicity-free space* if the Poisson brackets of any pair of invariant smooth functions vanish.

that there exists no Kähler structure on KT which is compatible with the symplectic form (by noticing that the first Betti number b_1(KT) of KT is 3). It follows that not only the symplectic action we have described in Example 3.4 is not Hamiltonian, but no other 2-torus action on KT is Hamiltonian either, since compact 4-manifolds with Hamiltonian 2-torus actions admit a Kähler structure compatible with the given symplectic form [15, 5].

Example 3.6 (Non-free symplectic action)**.** This is an example of a non--Hamiltonian, non-free symplectic 2-torus action on a compact, connected, symplectic 4-manifold. Consider the compact symplectic 4-manifold

$$(3.2) \qquad (M, \omega) := ((\mathbb{R}/\mathbb{Z})^2 \times S^2, \ \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\theta \wedge \mathrm{d}h).$$

There is a natural action of the 2-torus $T := \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ on expression (3.2), where the first circle of T acts on the first circle $\mathbb{R}/\mathbb{Z}$ of the left factor of M, and the right circle acts on S^2 by rotations (about the vertical axis); see Figure 3.1. This T-action is symplectic. However, it is not a Hamiltonian
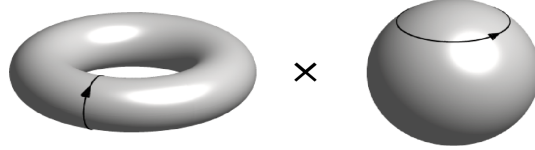

FIGURE 3.1. A non-free symplectic 2-torus action on $S^2 \times (\mathbb{R}/\mathbb{Z})^2$.

action because it does not have fixed points. It is also not free, because the stabilizer subgroup of a point (p, q), where q is the North or South pole of S^2, is a circle subgroup. In this case the principal orbits are the products of the circle orbits of the left factor $(\mathbb{R}/\mathbb{Z})^2$, and the circle orbits of the right factor (all orbits of the right factor are circles but the North and South poles, which are fixed points). Because these orbits are obtained by keeping the y-coordinate on the left factor constant, and the height on the right factor constant, $\mathrm{d}y = \mathrm{d}h = 0$, which implies that the product form vanishes along the principal orbits, which are Lagrangian, and hence coisotropic.

3.3. **Enumeration of all examples.** The examples in Section 3.2 are described by the following theorem.

Theorem 3.7 ([7])**.** *If a compact, connected, symplectic manifold M admits an effective symplectic T-action with a coisotropic orbit[2], then M is the total space of a fibration $F \hookrightarrow M \to B$ with base B being a torus bundle over a torus, and symplectic toric varietes F as fibers. On each of the toric varietes F a unique subtorus of T acts Hamiltonianly, any complement of which acts freely by permuting the toric varietes.*
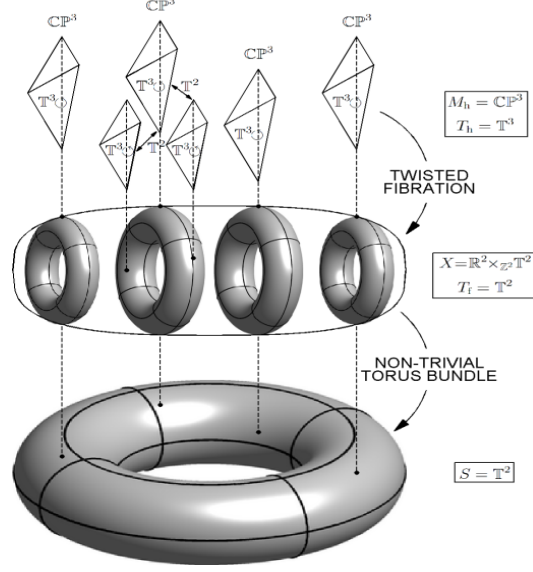
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[2]or Lagrangian orbit

FIGURE 3.2. A 10-dimensional symplectic manifold with a torus action with Lagrangian orbits. The fiber is the toric variety $(\mathbb{CP}^3, \mathbb{T}^3)$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The base is a non-trivial torus bundle $\mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$ over the 2-torus $\mathbb{T}^2$.

Let us describe specifically the model of the symplectic manifold (M, ω) with T-action in Theorem 3.7. The symplectic manifold (M, ω) is isomorphic (i.e. T-equivariantly symplectomorphic) to a fibration (see Figure 3.2) $M_\mathrm{h} \hookrightarrow G \times_H M_\mathrm{h} \to G/H$ with the following fiber and base. The fiber $(M_\mathrm{h}, T_\mathrm{h})$ is a symplectic-toric manifold (a toric variety). Here T_h is the maximal subtorus of T which acts on M in a Hamiltonian fashion. The base G/H is a (T/T_h)-bundle over a torus $(G/H)/T$, where G is (in general possibly) a non-abelian 2-step Nilpotent Lie group defined in terms of the Chern class of the principal torus bundle $M_\mathrm{reg} \to M_\mathrm{reg}/T$, and $H \leq G$ is a closed Lie subgroup of G defined in terms of the holonomy of a certain connection for the principal torus bundle $M_\mathrm{reg} \to M_\mathrm{reg}/T$.

Remark 3.8. Notice that (i) if the action is free, then the Hamiltonian subtorus T_h is trivial, and hence M is itself a torus bundle over a torus. Concretely, M is of the form G/H. The Kodaira variety (Example 3.4) is one of these spaces. Since M is a principal torus bundle over a torus, it is a nilmanifold for a two-step nilpotent Lie group as explained in Palais-Stewart [32]. In the case when this nilpotent Lie group is not abelian, M does not admit a Kähler structure, see Benson-Gordon [4]. (ii) In the case of 4-dimensional manifolds M, item (i) corresponds to the third case in Kodaira's description [22, Theorem 19] of the compact complex analytic surfaces which have a holomorphic $(2, 0)$-form that is nowhere vanishing,

see [9][3]. As mentioned, these were rediscovered by Thurston [39] as the first examples of compact connected symplectic manifolds without Kähler structure. (iii) If on the other hand the action is Hamiltonian, then $T_{\mathrm{h}} = T$, and in this case M is itself the toric variety (see [5, 15, 8] for the relations between symplectic toric manifolds and toric varieties). (iv) Henceforth, we may view the coisotropic orbit case as a twisted mixture of the Hamiltonian case, and of the free symplectic case[4]. The twist is determined by several symplectic invariants of the manifold, which we describe below.

3.4. **Symplectic ingredients in Theorem 3.7.** Before stating our second theorem on symplectic actions with coisotropic orbits we need to understand the construction of the maximal subtorus $T_{\mathrm{h}} \subset T$, the toric variety $M_{\mathrm{h}} \subset M$, as well as how T acts on M, and how G and H are constructed.

3.4.1. *Construction of T_{h}.* For this, we first need to know what our symplectic manifold with T-action looks like locally near the T-orbit of a point x. This is a consequence of the symplectic tube theorem of Benoist [2] and Ortega-Ratiu [29], which if one disregards the symplectic structure, says that there is an isomorphism of Lie groups ι from the stabilizer subgroup T_x onto some $\mathbb{T}^m$, an open $\mathbb{T}^m$-invariant neighborhood E_0 of the origin in $E = (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$, and a T-equivariant diffeomorphism $\Phi \colon K \times E_0 \to U$, where U is an open T-invariant neighborhood of x in M satisfying the condition $\Phi(1, 0) = x$. Here K is a complementary subtorus of the stabilizer subgroup T_x in T. For $t \in T$, let t_x and t_K be the unique elements in T_x and K, respectively, such that $t = t_x t_K$. We do not worry about what $\mathfrak{l}$ is just yet (it is a particular vector space), since the T-action does not affect it as we see next (we give a definition of $\mathfrak{l}$ in Section 3.4.3). The element $t \in T$ acts on $K \times (\mathfrak{l}/\mathfrak{h})^* \times \mathbb{C}^m$ as $t \cdot (k, \lambda, z) = (t_K k, \lambda, \iota(t_x) \cdot z)$. It follows that the stabilizer subgroup of (k, λ, z) is equal to the set of t_x in T_x such that $\iota(t_x)^j = 1$ for every j such that $z^j \neq 0$. There are 2^m different stabilizer subgroups T_y, $y \in U$. Since M is a compact manifold, is follows that there are only finitely many different stabilizer subgroups of T. The product of all the different stabilizer subgroups is a subtorus of T, which we denote by T_{h}, because the product of finitely many subtori is a compact and connected subgroup of T, and therefore it is a subtorus of T. One can show that the Hamiltonian subtorus T_{h} acts on M in a Hamiltonian fashion, and that any complementary subtorus T_{f} to T_{h} in T must acts freely on M.

3.4.2. *Construction of M_{h}.* One can show that there is a unique smooth distribution D which is integrable and T-invariant, and the integral manifolds of which are all $(2 \dim T_{\mathrm{h}})$-dimensional symplectic manifolds and isomorphic

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[3]In [9] the authors show that a compact connected symplectic 4-manifold with a symplectic 2-torus action admits an invariant complex structure and give an identification of those that do not admit a Kähler structure with Kodaira's class of complex surfaces which admit a nowhere vanishing holomorphic $(2, 0)$-form, but are not a torus or a K3 surface.

[4]The twist affects the manifold topology, the torus action, and the sympletic form.

to each other (i.e. T_h-equivariantly symplectomorphic to each other). We pick one of them and call it M_h. The symplectic form ω on M restricts to a symplectic form ω_h on M_h. On $(M_\mathrm{h}, \omega_\mathrm{h})$ the Hamiltonian torus T_h acts Hamiltonianly, and leaving it invariant.

3.4.3. *Definition of G.* Using the homotopy formula for the Lie derivative one can show that there is a unique antisymmetric bilinear form $\omega^\mathfrak{t}$ on $\mathfrak{t}$ such that

$$(3.3) \qquad \omega_x(X_M(x), Y_M(x)) = \omega^\mathfrak{t}(X, Y)$$

for every $X, Y \in \mathfrak{t}$ and every $x \in M$. Call $\mathfrak{l}$ the kernel of the antisymmetric bilinear form (the space $\mathfrak{l}$ is for example trivial if the principal orbits are Lagrangian, but this is not necessarily always the case). Call N the set of linear forms on $\mathfrak{l}$ which vanish on the Lie algebra $\mathfrak{t}_\mathrm{h}$ of T_h. Here we are assuming that $\mathfrak{t}_\mathrm{h} \subset \mathfrak{l}$, which is true because if $X \in \mathfrak{t}_x$, then $X_M(x) = 0$. By slightly abusing the notation, we write $N = (\mathfrak{l}/\mathfrak{t}_\mathrm{h})^*$. Set theoretically, we define $G = T \times N$. On G we define a non-standard group structure, in terms of the Chern class of $M_\mathrm{reg} \to M_\mathrm{reg}/T$, which one can naturally identify with an antisymmetric bilinear form $c \colon N \times N \to \mathfrak{l}$. We will wait to explain this identification to the next section, and we ask the reader to assume this for the time being, because now we can define the (in general) non-abelian operation on G

$$(3.4) \qquad (t, \zeta)\,(t', \zeta') = (t\,t'\,\mathrm{e}^{-c(\zeta, \zeta')/2}, \zeta + \zeta').$$

3.4.4. *Definition of H.* The subgroup H of G is defined as the set of $(t, \zeta) \in G$ such that $\zeta \in P$ and $t\,\tau_\zeta \in T_\mathrm{h}$. Here P is the so called *period lattice* of a natural group action on M/T (we explain more in Section 3.6.1). The elements $\tau_\zeta \in T$, $\zeta \in P$, encode the holonomy of a certain connection (Step 2 of Section 6) for the principal torus bundle $M_\mathrm{reg} \to M_\mathrm{reg}/T$.

3.4.5. *Definition of G/H.* The quotient G/H is done with respect to the non standard group structure in expression (3.4). Moreover, G/H is a nilmanifold. To go from $G \times_H M_\mathrm{h}$ to G/H we cancel the Hamiltonian action T_h, yet we still have an action of the free torus $T/T_\mathrm{h} \simeq T_\mathrm{f}$. The quotient G/H is a torus bundle $G/H \to (G/H)/(T/T_\mathrm{h})$ over a torus $(G/H)/(T/T_\mathrm{h})$, and it follows from work of Palais-Stewart that G/H is a nilmanifold for a 2-step nilpotent Lie group. In [7] we gave an explicit description of G/H as a nilmanifold.

3.4.6. *The T-action on M.* The T-action by translations on the left factor of G passes to an action on $G \times_H M_\mathrm{h}$. We have said that the torus T decomposes as $T = T_\mathrm{h}T_\mathrm{f}$, and T_h acts Hamiltonianly on each of the fibers, which are identified with the toric variety $(M_\mathrm{h}, \omega_\mathrm{h})$, leaving them invariants. The free torus T_f, which acts freely, simply permutes the fibers.

3.4.7. *The symplectic form on M.* The following formula bears no weight in the remaining of this paper. We write it here to convey that the classification we present is concrete. Let $\delta a = ((\delta t, \delta\zeta), \delta x)$, and $\delta'a = ((\delta't, \delta'\zeta), \delta'x)$ be tangent vectors to the product $G \times M_{\mathrm{h}}$ at the point $a = ((t, \zeta), x)$, where we identify each tangent space of T with $\mathfrak{t}$. Write $X = \delta t + c(\delta\zeta, \zeta)/2$ and $X' = \delta't + c(\delta'\zeta, \zeta)/2$. Let

$$
\begin{aligned}
\sigma_a(\delta a, \delta'a) \;=\; & \omega^{\mathfrak{t}}(\delta t, \delta't) + \delta\zeta(X'_{\mathfrak{l}}) - \delta'\zeta(X_{\mathfrak{l}}) - \mu(x)(c_{\mathrm{h}}(\delta\zeta, \delta'\zeta)) \\
& + (\omega_{\mathrm{h}})_x(\delta x, (X'_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) - (\omega_{\mathrm{h}})_x(\delta'x, (X_{\mathrm{h}})_{M_{\mathrm{h}}}(x)) \\
& + (\omega_{\mathrm{h}})_x(\delta x, \delta'x).
\end{aligned}
$$

Here X_{h} denotes the $\mathfrak{t}_{\mathrm{h}}$-component of $X \in \mathfrak{t}$ with respect to the decomposition $\mathfrak{t}_{\mathrm{h}} \oplus \mathfrak{t}_{\mathrm{f}}$, where $\mathfrak{t}_{\mathrm{h}}$ is the Lie algebra of T_{h}, and $\mathfrak{t}_{\mathrm{f}}$ is the Lie algebra of T_{f}. Similarly holds for $X_{\mathfrak{l}}$. The form ω_{h} is the restriction of ω to the chosen integral manifold M_{h}, cf. Section 3.4.2, and $c_{\mathrm{h}}(\delta\zeta, \delta'\zeta)$ is the $\mathfrak{t}_{\mathrm{h}}$-component of $c(\delta\zeta, \delta'\zeta) \in \mathfrak{l}$ in $\mathfrak{l} = \mathfrak{t}_{\mathrm{h}} \oplus (\mathfrak{l} \cap \mathfrak{t}_{\mathrm{f}})$.

If π_M denotes the canonical projection from $G \times M_{\mathrm{h}}$ onto $G \times_H M_{\mathrm{h}}$, then the T-invariant symplectic form on $G \times_H M_{\mathrm{h}}$ is the unique two-form β on $G \times_H M_{\mathrm{h}}$ such that $\sigma = \pi_M^* \beta$.

3.5. **Symplectic classification.** Theorem 3.7 gives a model of the manifold, the torus action, and the symplectic form, up to isomorphisms. But this is not a classification, because we do not know how many of these models appear, and, for the ones that do appear, we do not know whether they are isomorphic (T-equivariantly symplectomorphic). The first of these is the existence issue, while the second is the uniqueness issue.

Theorem 3.9 ([7]). *The compact, connected, symplectic manifolds (M, ω) with effective symplectic T-actions with some coisotropic orbit are determined (up to T-equivariant symplectomorphisms) by the following four symplectic invariants: 1) The antisymmetric bilinear form $\omega^{\mathfrak{t}} \colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$ defined as the restriction of ω to the T-orbits; 2) The Hamiltonian torus T_{h} and its associated momentum polytope Δ; 3) The period lattice P of the maximal the subgroup $N = (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ of $\mathfrak{l}^*$ acting on M/T, where $\mathfrak{l} := \ker(\omega^{\mathfrak{t}})$; 4a) The bilinear form $c \colon N \times N \to \mathfrak{l}$ encoding the Chern class of the bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$; 4b) An equivalence class $[\tau \colon P \to T]_{\exp(\mathcal{A})} \in \mathrm{Hom}_c(P, T)/\exp(\mathcal{A})$ encoding the holonomy of a certain connection for the bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$. Here $\mathrm{Hom}_c(P, T)$ is the set of maps $\tau \colon P \to T$, denoted by $\zeta \mapsto \tau_\zeta$, such that $\tau_{\zeta'}\tau_\zeta = \tau_{\zeta+\zeta'}\, \mathrm{e}^{c(\zeta', \zeta)/2}$, and $\exp(\mathcal{A})$ eliminates the dependance of the invariants on the choice of base point and on the particular choice of connection.*

For item 1) in Theorem 3.9 see formula (3.3). For item 2) see Section 3.4.1. For item 3) see Section 3.4.4. For item 4a) see Section 3.4.3.

Section 3.6 is devoted to a more explicit construction of the symplectic invariant 3), and to the construction and 4a), 4b) in Theorem 3.9. While

invariants 1) and 2) are relatively straightforward to define, 3) and 4) are more involved.

Remark 3.10. (a) Theorem 3.9 is a uniqueness theorem, i.e. we characterize that two manifolds are isomorphic if and only if they have the same invariants 1) –4). But we do not say which invariants 1) –4) do actually appear, which would be the existence theorem. (b) For simplicity, we have not stated this existence result here, but we shall say that, for example, any antisymetric bilinear form can appear as invariant 1), and any subtorus $S \subset T$ and Delzant polytope can appear as ingredient 2) etc. This is explained in [7]. (c) Theorem 3.9 is analogous to the well-known Delzant theorem [5], where Delzant proves that two manifolds are equivariantly symplectomorphic if and only is they have the same associated momentum polytope (uniqueness). Then he went on to prove that one can start from any abstract Delzant polytope and construct a manifold whose momentum polytope is precisely the one he started with (existence), and this is the part that we shall not write here.

3.6. **Symplectic invariants in Theorem 3.9.** We will explain what the invariants in the theorem are. Previously we explained invariants 1) and 2), and we said that later we would explain more about the natural group action on the orbit space (item 3)), the Chern class c, and the holonomy invariant (item 4)).

3.6.1. *Explanation of 3): group action on orbit space.* Using Leibniz identity for the Lie derivative one can show thar for each $X \in \mathfrak{t}$, $\widehat{\omega}(X) := -\mathrm{i}_{X_M}\omega$ is a closed basic one-form on M. For each $x \in M$, $\widehat{\omega}(X)_x$ is a linear form on $\mathrm{T}_x M$ which depends linearly on $X \in \mathfrak{t}$, and therefore $X \mapsto \widehat{\omega}(X)_x$ is an $\mathfrak{t}^*$-valued linear form on $\mathrm{T}_x M$, which we denote by $\widehat{\omega}_x$. In this way $x \mapsto \widehat{\omega}_x$ is an $\mathfrak{t}^*$-valued one-form on M, which we denote by $\widehat{\omega}$. With these conventions

$$\widehat{\omega}_x(v)(X) = \widehat{\omega}(X)_x(v) = \omega_x(v, X_M(x)), \quad x \in M, \ v \in \mathrm{T}_x M, \ X \in \mathfrak{t}.$$

Note that the $\mathfrak{t}^*$-valued one-form $\widehat{\omega}$ on M is basic and closed.

In the local model in Section 3.4.1 with $x \in M_{\mathrm{reg}}$, where $\mathfrak{h} = \mathfrak{t}_x = \{0\}$ and $m = 0$, at each point the $\mathfrak{t}^*$-valued one-form $\widehat{\omega}$ corresponds to the projection $(\delta t, \delta\lambda) \mapsto \delta\lambda : \mathfrak{t} \times \mathfrak{t}^* \to \mathfrak{t}^*$, and $\mathfrak{t} \times \{0\}$ is equal to the tangent space of the T-orbit. It follows that for every $p \in (M/T)_{\mathrm{reg}}$ the induced linear mapping $\widehat{\omega}_p : \mathrm{T}_p(M/T)_{\mathrm{reg}} \to \mathfrak{t}^*$ is a linear isomorphism. More generally, the strata for the T-action in $\overline{M} = K \times (\mathfrak{t}/\mathfrak{h})^* \times \mathbb{C}^m$ are of the form $\overline{M}^J$ in which J is a subset of $\{1, \ldots, m\}$ and $\overline{M}^J$ is the set of all (k, λ, z) such that $z^j = 0$ if and only if $j \in J$. If Σ is a connected component of the orbit type in M/T defined by the subtorus H of T with Lie algebra $\mathfrak{h}$, then for each $p \in \Sigma$ we have $\widehat{\omega}_p(X) = 0$ for all $X \in \mathfrak{h}$, and $\widehat{\omega}_p$ may be viewed as an element of $(\mathfrak{t}/\mathfrak{h})^* = \mathfrak{h}^0$, the set of all linear forms on $\mathfrak{t}$ which vanish on $\mathfrak{h}$. The linear mapping $\widehat{\omega}_p : \mathrm{T}_p\Sigma \to (\mathfrak{t}/\mathfrak{h})^*$ is a linear isomorphism.

Therefore an element $\zeta \in \mathfrak{l}^*$ acts naturally on $p \in (M/T)_{\mathrm{reg}}$ by traveling for time 1 from p in the direction that ζ points to. We denote the arrival point by $p + \zeta$. This action is in general not well defined at points in $(M/T) \setminus (M/T)_{\mathrm{reg}}$, it is only defined in the directions of vectors which as linear forms vanish on the stabilizer subgroup of the preimage under $\pi \colon M \to M/T$. Since T_{h} is the maximal stabilizer subgroup, and for each x, $T_x \subset T_{\mathrm{h}}$, the additive subgroup $N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$, viewed as the set of linear forms on $\mathfrak{l}$ which vanish on $\mathfrak{t}_{\mathrm{h}}$, is the maximal subgroup of the vector space $\mathfrak{l}^*$, which naturally acts on M/T. The invariant P is the period lattice for the N-action on M/T. This $\mathfrak{l}^*$-action turns M/T into what is called an $\mathfrak{l}^*$-*parallel space*. In [7, Section 11] one can find a classification of all V-*parallel spaces* for any vector space V. These spaces share common characteristics with both locally affine manifolds, and manifolds with corners. In [7] we prove that they are all isomorphic, as V-parallel spaces, to the product of a closed convex set and a torus. In the case that the V-parallel space is compact, this convex set is a convex polytope, and in the case of the $\mathfrak{l}^*$-parallel space being M/T, this convex polytope is moreover a Delzant polytope. A local analysis of the singularities of M/T allows us to define precisely the structure of $\mathfrak{l}^*$-parallel space.

Because of the global linear isomorphism $\widehat{\omega}_p \colon \mathrm{T}_p \Sigma \to (\mathfrak{l}/\mathfrak{h})^*$, any $\xi \in \mathfrak{l}^*$ may be viewed as a constant vector field on $(M/T)_{\mathrm{reg}}$. This is important later in the construction of the connection for the bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$, in terms of which in [7] the authors prove that M is isomorphic to $G \times_H M_{\mathrm{h}}$ (the sketch of proof of this is given in Section 6).

3.6.2. *Explanation of 4): Chern class and holonomy invariant.*

The holonomy invariant. A vector field L_ξ on M_{reg} is a *lift of ξ* if for all $x \in M_{\mathrm{reg}}$ we have that $\mathrm{T}_x \pi(L_\xi(x)) = \xi$. Linear assignments of lifts $\xi \in \mathfrak{l}^* \mapsto L_\xi$ depending linearly on ξ and connections for the principal torus bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ are objects that are equivalent. Most of the paper [7] is devoted to the construction of a nice connection

$$(3.5) \qquad\qquad \xi \in \mathfrak{l}^* \to L_\xi,$$

in terms of which one defines the model $G \times_H M_{\mathrm{h}}$, and the explicit isomorphism (i.e. T-equivariant symplectomorphism) between M and this model. By a "nice" connection we mean one for which the Lie brackets of two vector fields, and the symplectic pairings of two vector fields are particularly easy, and zero in most cases. Although the vector fields L_ξ are smooth on M_{reg}, many of them are singular (in a severe way, they blow up) on the lower dimensional strata of the orbit type stratification of M. We shall say more about this when we sketch the proof of Theorem 3.9 in Section 6.

In terms of the nice connection (3.5), in [7] we build what we call *the holonomy invariant*. We have to do this because the connection is not unique (although it is essentially unique up to a factor in the direction of $\mathfrak{t}$), and its holonomy depends on choices as well. Indeed, for each $\zeta \in P$ and $p \in M/T$,

the curve $\gamma_\zeta(t) := p + t\,\zeta$, where $0 \leq t \leq 1$, is a loop in M/T. If $x \in M$ and $p = \pi(x)$, then the curve $\delta(t) = \mathrm{e}^{t\,L_\zeta}(x)$, $0 \leq t \leq 1$, is called the *horizontal lift of* γ_ζ which starts at x, because $\delta(0) = x$, and $\delta'(t) = L_\zeta(\delta(t))$ is a horizontal tangent vector which is mapped by $\mathrm{T}_{\delta(t)}\pi$ to the constant vector ζ. This implies that $\pi(\delta(t)) = \gamma_\zeta(t)$ for all $0 \leq t \leq 1$. The element of T which maps $\delta(0) = x$ to $\delta(1)$ is called the *holonomy* $\tau_\zeta(x)$ of the loop γ_ζ at x with respect to the connection (3.5). Because $\delta(1) = \mathrm{e}^{L_\zeta}(x)$, we have that $\tau_\zeta(x) \cdot x = \mathrm{e}^{L_\zeta}(x)$. The element $\tau_\zeta(x)$ does depend on the point $x \in M$, on the period $\zeta \in P$, or on the choice of nice connection. Hence why we have to construct a more refined invariant, the equivalence class of τ under a Lie subgroup which we denote by $\exp(\mathcal{A})$, and which eliminates the dependance on both the choice of connection and the choice of base point.

The Chern Class. As discussed in Section 3.6.1, M/T is isomorphic to the product of a Delzant polytope Δ and a torus S. In fact S is equal to N/P, where recall that N is the largest group which acts naturally on the orbit space M/T, and P is its period lattice. Also, $(M/T)_{\mathrm{reg}} \simeq \Delta^{\mathrm{int}} \times (N/P)$, where Δ^{int} denotes the interior of the Delzant polytope Δ. Any connection for the principal T-bundle $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ has a *curvature form*, which is a smooth $\mathfrak{t}$-valued two-form on the quotient M_{reg}/T. The cohomology class of this curvature form in an element of $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$, which is independent of the choice of the connection. The N-action on M/T leaves $M_{\mathrm{reg}}/T \simeq (M/T)_{\mathrm{reg}}$ invariant, with orbits isomorphic to the torus N/P. The pull-back to the N-orbits defines an isomorphism from $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ onto $\mathrm{H}^2(N/P, \mathfrak{t})$, which in turn is identified[5] with $(\Lambda^2 N^*) \otimes \mathfrak{t}$. It follows from the proof in which we construct the nice connection in terms of which we construct the model of M (cf. Section 6 for a sketch), that $c \colon N \times N \to \mathfrak{l}$, viewed as an element in $c \in (\Lambda^2 N^*) \otimes \mathfrak{l} \subset (\Lambda^2 N^*) \otimes \mathfrak{t}$, equals the negative of the pull-back to an N-orbit of the cohomology class of the curvature form. This shows that $c \colon N \times N \to \mathfrak{l}$ is independent of the choice of free subtorus T_{f}. The Chern class $\mathcal{C}$ of the principal T-bundle $\pi \colon M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ is an element of $\mathrm{H}^2(M_{\mathrm{reg}}/T, T_{\mathbb{Z}})$. It is known that the image of $\mathcal{C}$ in $\mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ under the coefficient homomorphism $\mathrm{H}^2(M_{\mathrm{reg}}/T, T_{\mathbb{Z}}) \to \mathrm{H}^2(M_{\mathrm{reg}}/T, \mathfrak{t})$ is equal to the negative of the cohomology class of the curvature form of any connection in the principal T-bundle, and hence c represents $\mathcal{C}$.

3.6.3. *Symplectic invariants of the Kodaira variety.* Recall the Kodaira variety $M = \mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2$ in Example 3.4 (see also Figure 3.3). In this case $T = (\mathbb{R}/\mathbb{Z})^2$, $\mathfrak{t} \simeq \mathbb{R}^2$ and its invariants are: 1) the trivial antisymmetric bilinear form: $\omega^{\mathfrak{t}} = 0$. 2) The trivial Hamiltonian torus: $T_{\mathrm{h}} = \{[0, 0]\}$ and the Delzant polytope consisting only of the origin: $\Delta = \{(0, 0)\}$; 3) the period lattice is $P = \mathbb{Z}^2$; 4a) the bilinear form c representing the Chern

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[5]as already observed by Élie Cartan.

class is given by $c \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, defined by $c(e_1, e_2) = e_1$. 4b) The holonomy invariant is the class of τ given by $\tau_{e_1} = \tau_{e_2} = [0, 0]$. In this case
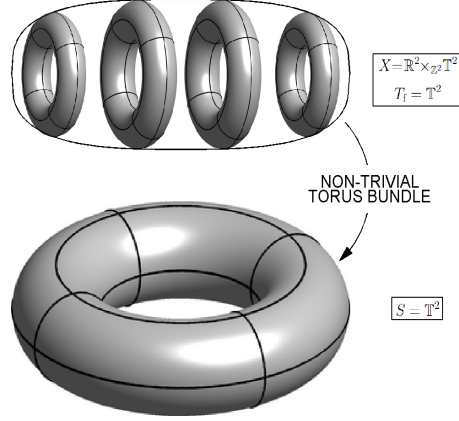

FIGURE 3.3. The Kodaira variety endowed with a free symplectic 2-torus action with Lagrangian orbits.

$G = (\mathbb{R}/\mathbb{Z})^2 \times \mathbb{R}^2$, $M_{\mathrm{h}} = \{p\}$, and $H = \{[0, 0]\} \times \mathbb{Z}^2$. The model of M is $G \times_H M_{\mathrm{h}} \simeq G/H \simeq \mathbb{R}^2 \times_{\mathbb{Z}^2} (\mathbb{R}/\mathbb{Z})^2$.

4. SYMPLECTIC TORUS ACTIONS WITH MAXIMAL SYMPLECTIC ORBITS

This section is based on parts of [33]. Let (M, ω) be a compact, connected, $2n$-dimensional symplectic manifold. Let T be a torus. Suppose that T acts effectively and symplectically on M.

Symplectic orbit condition. *Throughout this section we assume that there exists a $\dim T$-dimensional orbit which is a symplectic submanifold.*

4.1. **The meaning of the symplectic condition.** A submanifold C of the symplectic manifold (M, ω) is *symplectic* if the restriction $\omega|_C$ of the symplectic form ω to C is symplectic. Let $\mathfrak{t}_x$ denote the Lie algebra of T_x. In Section 3.4.3 we observed that $\mathfrak{t}_x \subset \ker \omega^{\mathfrak{t}}$. Since the antisymmetric bilinear form $\omega^{\mathfrak{t}}$ introduced therein (which gives the restriction of ω to the T-orbits) is non-degenerate because of the symplectic condition, its kernel $\ker \omega^{\mathfrak{t}}$ is trivial. Hence $\mathfrak{t}_x$ is the trivial vector space, and hence T_x, which is a closed and hence compact subgroup of T, must be a finite group (which is of course abelian since T is abelian). Since T is compact, the action of T on M is a proper action, the mapping $t \mapsto t \cdot x \colon T/T_x \to T \cdot x$ is a diffeomorphism, and, in particular, the dimension of the quotient group T/T_x equals the dimension of $T \cdot x$. Since each T_x is finite, the dimension of T/T_x equals $\dim T$, and hence every T-orbit is $\dim T$-dimensional. Since the symplectic form ω restricted to any T-orbit of the T-action is non-degenerate, $T \cdot x$ is a symplectic submanifold of (M, ω). Therefore, we can conclude that at least one T-orbit is a $\dim T$-dimensional symplectic

submanifold of the symplectic manifold (M, ω) if and only if every T-orbit is a $\dim T$-dimensional symplectic submanifold of (M, ω).

Then there exists only finitely many different subgroups of T which occur as stabilizer subgroups of the action of T on M, and each of them is a finite group. Indeed, we know that every stabilizer subgroup of the action of T on M is a finite group. It follows from the tube theorem of Koszul (cf. [24] or [6, Theorem 2.4.1]), that in the case of a compact smooth manifold equipped with an effective action of a torus T, there exists only finitely many different subgroups of T which occur as stabilizer subgroups.

Remark 4.1. In the case when $\dim M = 4$ and $\dim T = 2$, the stabilizer subgroup of the T-action at every point in M is a cyclic abelian group (cf. [33, Lemma 2.2.6]). This follows from the fact that a finite group acting linearly on a disk must be a cyclic group acting by rotations (after application of the symplectic tube theorem).

4.2. **Examples.**

Example 4.2 (Free symplectic action)**.** The 4-dimensional torus $M := (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/\mathbb{Z})^2$ endowed with the standard symplectic form, on which the 2-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ acts by multiplications on two of the copies of $\mathbb{R}/\mathbb{Z}$ inside of $(\mathbb{R}/\mathbb{Z})^4$, is symplectic manifold with symplectic orbits (see Figure 4.1). The T-orbits are symplectic 2-tori, and $M/T = (\mathbb{R}/\mathbb{Z})^2$.
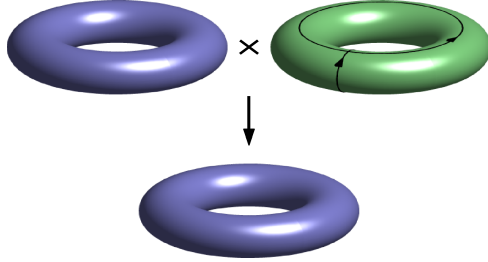

FIGURE 4.1. A 4-dimensional torus $(\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{R}/\mathbb{Z})^2$ endowed with a symplectic 2-torus action with symplectic orbits on the right factor. The orbit space is $(\mathbb{R}/\mathbb{Z})^2$.

Example 4.3 (Free symplectic action)**.** Let $(M, \omega) := S^2 \times (\mathbb{R}/\mathbb{Z})^2$ endowed with the product symplectic form (of the standard symplectic (area) form on $(\mathbb{R}/\mathbb{Z})^2$ and the standard area form on S^2). Let $T := (\mathbb{R}/\mathbb{Z})^2$ act on M by translations on the right factor. This is a free action on M the orbits of which are symplectic 2-tori. In this case $M/T = S^2$.

Example 4.4 (Non-free symplectic action)**.** Let $P := S^2 \times (\mathbb{R}/\mathbb{Z})^2$ equipped with the product symplectic form of the standard symplectic (area) form on S^2 and the standard area form on the sphere $(\mathbb{R}/\mathbb{Z})^2$. The 2-torus $T := (\mathbb{R}/\mathbb{Z})^2$ acts freely by translations on the right factor of P, see Figure 4.2.
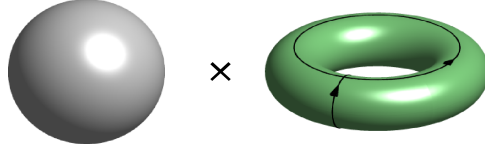
FIGURE 4.2. A free symplectic action of a 2-torus on the right factor of the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$.

Let the finite group $\mathbb{Z}/2\,\mathbb{Z}$ act on S^2 by rotating each point horizontally by 180 degrees, and let $\mathbb{Z}/2\,\mathbb{Z}$ act on $(\mathbb{R}/\mathbb{Z})^2$ by the antipodal action on the first circle $\mathbb{R}/\mathbb{Z}$. The diagonal action of $\mathbb{Z}/2\,\mathbb{Z}$ on P is free. Therefore, the quotient space $S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ is a smooth manifold. Let $M := S^2 \times_{\mathbb{Z}/2\,\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$ be endowed with the symplectic form ω and T-action inherited from the ones given in the product $S^2 \times (\mathbb{R}/\mathbb{Z})^2$, where $T = (\mathbb{R}/\mathbb{Z})^2$. The action of T on M is not free, and the T-orbits are symplectic 2-dimensional tori. The orbit space M/T is $S^2/(\mathbb{Z}/2\,\mathbb{Z})$, which is a smooth orbifold with two singular points of order 2, the South and North poles of S^2.

4.3. **Enumeration of all examples.** The following provides an enumeration of all examples that can appear.

Theorem 4.5 ([33])**.** *If a compact, connected, symplectic manifold M admits an effective symplectic T-action with a $\dim T$-dimensional symplectic orbit, then M is isomorphic (i.e. T-equivariantly symplectomorphic) to an orbifold bundle $\widetilde{M/T} \times_\Gamma T$ where $\widetilde{M/T}$ is the orbifold universal cover[6] of M/T and $\Gamma := \pi_1^{\mathrm{orb}}(M/T)$, Γ acts on T by means of the monodromy $\mu \colon \Gamma \to T$ of the flat connection $\{(\mathrm{T}_x(T \cdot x))^{\omega_x}\}_{x \in M}$ of symplectic orthogonal complements to the tangent spaces to the T-orbits, and Γ acts on the Cartesian product $\widetilde{M/T} \times T$ by the diagonal action $x\,(y,\,t) = (x \star y^{-1},\, \mu(x) \cdot t)$, where $\star \colon \Gamma \times \widetilde{M/T} \to \widetilde{M/T}$ denotes the natural action of Γ on $\widetilde{M/T}$. The T-action and symplectic form on $\widetilde{M/T} \times_\Gamma T$ are inherited from the T-action on the right factor of $\widetilde{M/T} \times T$ and the natural product form.*

Remark 4.6. The article [10] shows that the first Betti number of M/T is equal to the first Betti number of M minus the dimension of T.

4.4. **Ingredients of Theorem 4.5.**

4.4.1. *M/T is a smooth orbifold.* We denote by $\pi \colon M \to M/T$ the canonical projection $\pi(x) := T \cdot x$. The orbit space M/T is endowed with the maximal topology for which π is continuous (which is a Hausdorff topology). Since M is compact and connected, M/T is compact and connected. If C is a connected component of an orbit type $M^H := \{x \in M \mid T_x = H\}$, the action of T on C induces a free and proper action of the quotient torus T/H on

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[6]The universal cover and the fundamental group must be based at the same point p_0.

C, and the image $\pi(C)$ has a unique smooth manifold structure for which $\pi \colon C \to \pi(C)$ is a principal T/H-bundle.

Remark 4.7. In general, M/T is not a smooth manifold, cf. Example 4.4.

Our next goal is to explain how M/T has a natural smooth orbifold structure. By the tube theorem (see for instance [6, Theorem 2.4.1]) if $x \in M$ there is a T-invariant open neighborhood U_x of $T \cdot x$ and a T-equivariant diffeomorphism $\Phi_x \colon U_x \to T \times_{T_x} D_x$, where D_x is an open disk centered at the origin in $\mathbb{C}^{k/2}$, $k := \dim M - \dim T$. Here the stabilizer T_x acts by linear transformations on the disk D_x. The action of T on $T \times_{T_x} D_x$ is induced by the translational action of T on the left factor of the product $T \times D_x$. The T-equivariant diffeomorphism Φ_x induces a homeomorphism $\widehat{\Phi}_x \colon D_x/T_x \to \pi(U_x)$, and we have a commutative diagram

$$(4.1) \qquad \begin{array}{ccccc} T \times D_x & \xrightarrow{\ \pi_x\ } & T \times_{T_x} D_x & \xrightarrow{\ \Phi_x\ } & U_x \\[2pt] {\scriptstyle i_x}\big\uparrow & & {\scriptstyle p_x}\big\downarrow & & \big\downarrow{\scriptstyle \pi|_{U_x}} \\[2pt] D_x & \xrightarrow{\ \pi'_x\ } & D_x/T_x & \xrightarrow{\ \widehat{\Phi}_x\ } & \pi(U_x) \end{array}$$

In this diagram π_x, π'_x, p_x are the canonical projections and i_x is the inclusion. Let $\phi_x := \widehat{\Phi}_x \circ \pi'_x$. In [33] it is shown that the collection $\widehat{\mathcal{A}} := \{(\pi(U_x), D_x, \phi_x, T_x)\}_{x \in M}$ is an orbifold atlas for M/T. We call $\mathcal{A}$ the class of atlases equivalent to $\widehat{\mathcal{A}}$. We simply write M/T for M/T endowed with the class $\mathcal{A}$.

4.4.2. *The connection* $\Omega := \{(\mathrm{T}_x(T \cdot x))^{\omega_x}\}_{x \in M}$ *is flat.* The collection of subspaces $\Omega := \{\Omega_x\}_{x \in M}$ where $\Omega_x := (\mathrm{T}_x(T \cdot x))^{\omega_x}$, is a smooth distribution on M and $\pi \colon M \to M/T$ is a smooth principal T-bundle of which Ω is a T-invariant flat connection. To prove this one uses the diagram (4.1), the construction of the orbifold structure on the orbit space M/T, and the symplectic tube theorem (cf. Benoist [2, Prop. 1.9] and Ortega and Ratiu [29]). Put it differently, Ω is a smooth T-invariant $(\dim M - \dim T)$-dimensional foliation of M.

4.4.3. M/T *is a good orbifold, and hence* $\widetilde{M/T}$ *is a manifold.* Let $\mathcal{I}_x$ be the maximal integral manifold of Ω. The inclusion $i_x \colon \mathcal{I}_x \to M$ is an injective immersion and $\pi \circ i_x \colon \mathcal{I}_x \to M/T$ is an orbifold covering . Since $\widetilde{M/T}$ covers any covering of M/T, it covers $\mathcal{I}_x$, which is a manifold. Because a covering of a manifold is itself a manifold, the universal cover $\widetilde{M/T}$ is a manifold. Readers unfamiliar with orbifolds may consult [33, Section 9].

4.4.4. *The orbifold fundamental group* Γ. Let $\mathcal{O}$ be a connected (smooth) orbifold; the case we are interested in is $\mathcal{O} = M/T$, but the following notions hold in general. The *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(\mathcal{O}, x_0)$ *of* $\mathcal{O}$ *based at* x_0 is the set of homotopy classes of orbifold loops that have the same initial and end point x_0. The operation on this set is the classical composition law

by concatenation of loops. The set $\pi_1^{\mathrm{orb}}(\mathcal{O}, x_0)$ endowed with this operation is a group. It is an exercise to check (as in the classical case) that changing the base point results in an isomorphic group. Thurston proved that any connected (smooth) orbifold $\mathcal{O}$ posseses an orbifold covering[7] $\widetilde{\mathcal{O}}$ which is universal (i.e. it covers any other cover), and the orbifold fundamental group of which based at any regular point is trivial.

4.4.5. *T-action on M.* The T-action on $\widetilde{M/T} \times_\Gamma T$ is the T- action inherited from the action of T by translations on the right factor of $\widetilde{M/T} \times T$.

4.4.6. *Symplectic form on M.* The orbit space M/T, which we have seen is a smooth orbifold, comes endowed with a symplectic structure. In order to define it, we review the notion of symplectic form on a smooth orbifold $\mathcal{O}$. A *(smooth) differential form* σ (respectively a *symplectic form*) on $\mathcal{O}$ is a collection $\{\widetilde{\sigma}_i\}$, in which $\widetilde{\sigma}_i$ is a Γ_i-invariant differential form (respectively a symplectic form) on each $\widetilde{U}_i$, and such that any two of these forms coincide on overlaps (by what we mean that for each $x \in \widetilde{U}_i$, $y \in \widetilde{U}_j$ with $\phi_i(x) = \phi_j(y)$, there is a diffeomorphism ψ from a neighborhood of x to a neighborhood of y such that $\phi_j \circ \psi = \phi_i$ on it, and $\psi^* \sigma_j = \sigma_i$). A *symplectic orbifold* is a pair consisting of smooth orbifold and a symplectic form on it.

Let σ be a differential form on an orbifold $\mathcal{O}$ given by $\{\widetilde{\sigma}_i\}$, where each $\widetilde{\sigma}_i$ is a Γ_i-invariant differential form on $\widetilde{U}_i$. If $f \colon \mathcal{O}' \to \mathcal{O}$ is an orbifold diffeomorphism, the *pull-back* of σ is the unique differential form σ' on $\mathcal{O}'$ given by $\widetilde{f}^* \widetilde{\sigma}_i$ on each $\widetilde{f}^{-1}(\widetilde{U}_i)$. We shall use the notation $\sigma' := f^* \sigma$. Similarly one defines the pullback under a principal T-orbibundle $p \colon Y \to X/T$ of σ. The symplectic orbifolds $(\mathcal{O}_1, \nu_1)$, $(\mathcal{O}_2, \nu_2)$ are *symplectomorphic* if there exists an orbifold diffeomorphism $f \colon \mathcal{O}_1 \to \mathcal{O}_2$ with $f^* \nu_2 = \nu_1$ (analogously to the case of manifolds, the map f is called an *orbifold symplectomorphism*).

In our case of $\mathcal{O} = M/T$, one can show that there exists a unique 2-form ν on M/T such that $\pi^* \nu|_{\Omega_x} = \omega|_{\Omega_x}$ for every $x \in M$. Moreover, ν is a symplectic form. Therefore $(M/T, \nu)$ is a compact, connected, symplectic orbifold. The symplectic form on $\widetilde{M/T}$ is the pullback by the covering map $\widetilde{M/T} \to M/T$ of ν and the symplectic form on T is the unique T-invariant symplectic form determined by ω^{t}. The symplectic form on $\widetilde{M/T} \times T$ is the product symplectic form. The symplectic form on $\widetilde{M/T} \times_\Gamma T$ is induced on the quotient by the product form.

4.5. Symplectic classification. The following is the classification theorem, up to T-equivariant symplectomorphisms. While in Theorem 4.5 we make no assumption on the dimension of T, in order to give a classification we need to assume that $\dim T = \dim M - 2$. The reason is that in this case the orbit space M/T is 2-dimensional and 2-dimensional orbifolds are classified (by Thurston), unlike their higher dimensional analogues.

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[7]a unique up to equivalence.

Theorem 4.8 ([33])**.** *The compact, connected, symplectic $2n$-dimensional manifolds (M, ω) equipped with an effective symplectic T-action with some $(2n-2)$-dimensional symplectic T-orbit are classified by the following four symplectic invariants:* 1) *The non-degenerate antisymmetric bilinear $\omega^{\mathfrak{t}}\colon \mathfrak{t} \times \mathfrak{t} \to \mathbb{R}$: the restriction of the symplectic form ω to the T-orbits;* 2) *The Fuchsian signature $(g; \vec{o})$ encoding the smooth orbisurface M/T;* 3) *The symplectic area λ of M/T;* 4) *The monodromy invariant of the connection Ω of symplectic orthocomplements to the tangent spaces to the T-orbits:*

$$\mathcal{G}_{(g, \vec{o})} \cdot ((\mu_{\mathrm{h}}(\alpha_i),\, \mu_{\mathrm{h}}(\beta_i))_{i=1}^g,\ (\mu_{\mathrm{h}}(\gamma_k))_{k=1}^m) \in T^{2g+m}/\mathcal{G}_{(g, \vec{o})},$$

where

$$\mathcal{G}_{(g;\vec{o})} := \left\{ \left(\begin{array}{cc} A & 0 \\ C & D \end{array} \right) \ \Big|\ A \in \mathrm{Sp}(2g,\, \mathbb{Z}),\ \ D \in \mathcal{MS}_m^{\vec{o}} \right\},$$

the map $\mu_{\mathrm{h}}\colon \mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z}) \to T$ is the monodromy homomorphism of Ω, the collection $\{\alpha_i, \beta_i\} \subset \mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ is a symplectic basis of a maximal free subgroup, and $\{\gamma_k\} \subset \mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$ is a geometric torsion basis.

Remark 4.9. Theorem 4.8 is a uniqueness theorem, i.e. we characterize that two manifolds are isomorphic (T-equivariantly symplectomorphic) if and only if they have the same symplectic invariants 1)-4). Theorem 4.8 does not say which symplectic invariants 1)-4) appear; such a statement would be given by an existence theorem. In order to keep the presentation simple and the paper succinct, we do not state the aforementioned existence result here, but we shall say that, for instance, any non-degenerate antisymetric bilinear form can appear as symplectic invariant 1), and any tuple $(g; \vec{o})$ such that $(g; \vec{o})$ is not of the form $(0; o_1)$ or of the form $(0; o_1, o_2)$ with $o_1 < o_2$ also shows up. This is explained in [33].

Remark 4.10. Theorem 4.8 is analogous to the uniqueness part of Delzant's Theorem ([5]) which states that two Delzant manifolds (also known as symplectic-toric manifolds) are isomorphic (i.e. T-equivariantly symplecto-morphic) if and only is they have the same associated momentum polytope (uniqueness), which is a particular type of polytope known as a *Delzant poly-tope*[8]. Then Delzant went on to proving that one can start from any abstract Delzant polytope and construct a symplectic manifold with a torus action whose momentum polytope is precisely the one he started with (existence); and this is the part that we shall not write here.

4.6. **Symplectic invariants in Theorem 4.8.** Ingredient 1) was explained in Section 3. Ingredients 2) and 3) are easy to explain, which we do briefly below. We focus on describing ingredient 4).

4.6.1. *Invariant 2): the orbifold invariant.* Let $\mathcal{O}$ be a smooth orbisurface with m cone points $p_1, \ldots, p_m$. The *Fuchsian signature of $\mathcal{O}$* is $(g; \vec{o})$ where g stands for the genus of the surface underlying $\mathcal{O}$, $o_k > 0$ is the order of p_k,

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[8]Simple, rational, and smooth polytope

and $\vec{o} = (o_1, \ldots, o_m)$, where $o_k \leq o_{k+1}$ for all $1 \leq k \leq m - 1$. We denote by $\text{sig}(\mathcal{O})$ the *Fuchsian signature of* $\mathcal{O}$. The orbit space M/T, which is a smooth orbisurface, is topologically classified by $(g; \vec{o})$.

4.6.2. *Invariant 3): the area invariant.* The symbol $\int_{\mathcal{O}} \sigma$ denotes the integral of a differential form σ on the orbifold $\mathcal{O}$. If $(\mathcal{O}, \sigma)$ is a symplectic orbifold, $\int_{\mathcal{O}} \sigma$ is the *total symplectic area of* $(\mathcal{O}, \sigma)$. By the orbifold Moser's theorem [26, Theorem 3.3], if (M, ω), (M', ω') are compact, connected, symplectic manifolds equipped with an action of a torus T of dimension $\dim T = \dim M - 2$, for which the T-orbits are $\dim T$-dimensional symplectic submanifolds of (M, ω), and such that $\int_{M/T} \nu = \int_{M'/T} \nu'$ and $\text{sig}(M/T) = \text{sig}(M'/T)$, then $(M/T, \nu), (M'/T, \nu')$ are symplectomorphic.

4.6.3. *Invariant 4): the monodromy invariant.* Let g, n, o_k, $1 \leq k \leq m$, be non-negative integers. Let Σ be a compact, connected, orientable smooth orbisurface, with underlying topological space a surface of genus g with m singular points $p_1, \ldots, p_m$ of orders o_k. The orbifold fundamental group is $\pi_1^{\text{orb}}(\Sigma, p_0) = \langle \{\alpha_i, \beta_i\}_{i=1}^g, \{\gamma_k\}_{k=1}^m | \prod_{k=1}^m \gamma_k = \prod_{i=1}^g [\alpha_i, \beta_i], \gamma_k^{o_k} = 1, 1 \leq k \leq m \rangle$, where α_i, β_i, $1 \leq i \leq g$ is a symplectic basis of the surface underlying Σ, and the γ_k are the homotopy classes of small loops $\widetilde{\gamma}_k$ around the orbifold singular points. By abelianizing $\pi_1^{\text{orb}}(\Sigma, p_0)$ we get $\text{H}_1^{\text{orb}}(\Sigma, \mathbb{Z}) = \langle \{\alpha_i, \beta_i\}_{i=1}^g, \{\gamma_k\}_{k=1}^m \mid \sum_{k=1}^n \gamma_k = 0, o_k \gamma_k = 0, 1 \leq k \leq m \rangle$.

Example 4.11. Let Σ be the smooth orbisurface with underlying space $(\mathbb{R}/\mathbb{Z})^2$ and with exactly one cone point, say p_1, of order $o_1 = 2$. Let $\widetilde{\gamma}$ be a boundary loop of a small disk containing p_1, and let $\gamma = [\widetilde{\gamma}]$. Let α, β define the standard basis of loops corresponding to $(\mathbb{R}/\mathbb{Z})^2$, that is, α, β are a basis of the quotient free first orbifold homology group of Σ. For $x_0 \in \Sigma$, $\pi_1^{\text{orb}}(\Sigma, x_0) = \langle \alpha, \beta, \gamma \,|\, [\alpha, \beta] = \gamma, \gamma^2 = 1 \rangle$, and $\text{H}_1^{\text{orb}}(\Sigma, \mathbb{Z}) = \langle \alpha, \beta, \gamma \,|\, \gamma = 1, 2\gamma = 0 \rangle \simeq \langle \alpha, \beta \rangle$.

Fix an m-dimensional vector of strictly positive integers $\vec{o}$ and a non-negative integer g. Let $\mathcal{MS}_m^{\vec{o}}$ be the set of matrices $B \in \text{GL}(m, \mathbb{Z})$ such that $B \cdot \vec{o} = \vec{o}\}$. Define the quotient $T_{(g; \vec{o})}^{2g+m}/\mathcal{G}_{(g, \vec{o})}$ where $T_{(g; \vec{o})}^{2g+n}$ is the set of points $(t_i)_{i=1}^{2g+m} \in T^{2g+m}$ such that the order of t_i is a multiple of o_i for all $2g + 1 \leq i \leq m$, and

$$\mathcal{G}_{(g; \vec{o})} := \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in \text{GL}(2g + m, \mathbb{Z}) \,|\, A \in \text{Sp}(2g, \mathbb{Z}), \ D \in \mathcal{MS}_m^{\vec{o}} \right\}.$$

We call $T_{(g; \vec{o})}^{2g+m}/\mathcal{G}_{(g, \vec{o})}$ the *Fuchsian signature* space associated to $(g; \vec{o})$

In the case of our compact, connected symplectic manifold (M, ω) with an effective symplectic T-action with $\dim T$-dimensional symplectic T-orbits we have the following ingredients: let $(g; \vec{o}) \in \mathbb{Z}^{1+m}$ be the Fuchsian signature of M/T; let $\{\gamma_k\}_{k=1}^m$ be a basis of small loops around the cone points $p_1, \ldots, p_n$ of M/T, viewed as an orbifold; let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a symplectic basis

of a *free* subgroup F of the orbifold homology group $\mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$, whose direct sum with the *torsion* subgroup is $\mathrm{H}_1^{\mathrm{orb}}(M/T, \mathbb{Z})$; let μ_{h} be the homomomomorphism induced on homology by the monodromy homomorphism μ of the connection Ω of symplectic orthogonal complements to the tangent spaces to the T-orbits. The *monodromy invariant of* (M, ω, T) is the $\mathcal{G}_{(g;\vec{o})}$-orbit

$$(4.2) \quad \mathcal{G}_{(g,\vec{o})} \cdot ((\mu_{\mathrm{h}}(\alpha_i), \mu_{\mathrm{h}}(\beta_i))_{i=1}^g, (\mu_{\mathrm{h}}(\gamma_k))_{k=1}^m) \in T_{(g;\vec{o})}^{2g+m}/\mathcal{G}_{(g,\vec{o})}.$$

Even though (4.2) depends on choices, one can show that it is well-defined.

4.6.4. *Invariants of* $M = S^2 \times_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{R}/\mathbb{Z})^2$. The invariants are: the non-degenerate antisymmetric bilinear form $\omega^{\mathbb{R}^2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; 2) The Fuchsian signature $(g; \vec{o}) = (0; 2, 2)$ of the orbit space $M/\mathbb{T}^2$; 3) The symplectic area of $S^2/(\mathbb{Z}/2\mathbb{Z})$: 1 (half of the area of S^2); 4) The monodromy invariant:

$$\mathcal{G}_{(0;2,2)} \cdot (\mu_{\mathrm{h}}(\gamma_1), \mu_{\mathrm{h}}(\gamma_2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \cdot ([1/2, 0], [1/2, 0]).$$

Here the γ_1, γ_2 are small loops around the poles of S^2. Then $M/T = S^2/(\mathbb{Z}/2\mathbb{Z})$, $\pi_1^{\mathrm{orb}}(M/T, p_0) = \langle \gamma_1 \,|\, \gamma_1^2 = 1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$, and $\mu \colon \langle \gamma_1 \,|\, \gamma_1^2 = 1 \rangle \to T = (\mathbb{R}/\mathbb{Z})^2$ is $\mu(\gamma_1) = [1/2, 0]$. We have a T-equivariant symplectomorphism

$$\widetilde{M/T} \times_{\pi_1^{\mathrm{orb}}(M/T, p_0)} T = \widetilde{S^2/(\mathbb{Z}/2\mathbb{Z})} \times_{\pi_1^{\mathrm{orb}}(S^2/(\mathbb{Z}/2\mathbb{Z}), p_0)} (\mathbb{R}/\mathbb{Z})^2 \simeq M.$$

5. Symplectic 2-torus actions on 4-manifolds

Our only assumption now is that T is a 2-dimensional torus and (M, ω) is a compact, connected 4-dimensional symplectic manifold on which T acts effectively and symplectically. The following is a simplified version of the main result of [33]; readers can consult [33, Theorem 8.2.1] for the most informative and complete version of the statement.

Theorem 5.1 ([33]). *If (M, ω) is compact, connected, symplectic 4-manifold equipped with an effective symplectic action of a 2-torus T then one and only one of the following four cases occurs.*

1) *(M, ω) is a symplectic toric 4-manifold.*
2) *(M, ω) is equivariantly symplectomorphic to $(\mathbb{R}/\mathbb{Z})^2 \times S^2$, where on $(\mathbb{R}/\mathbb{Z})^2 \times S^2$ the symplectic form is a product form. The action of the 2-torus is: one circle acts on the first circle of $(\mathbb{R}/\mathbb{Z})^2$ by translations, while the other circle acts on S^2 by rotations about the vertical axes.*
3) *(M, ω) is equivariantly symplectomorphic to $(T \times \mathfrak{t}^*)/\iota(P)$, where $T \times \mathfrak{t}^*$ is equipped with the standard cotangent bundle form and the standard T-action on left factor of $T \times \mathfrak{t}^*$, both of which descend to $(T \times \mathfrak{t}^*)/\iota(P)$, $P \subset \mathfrak{t}^*$ is a discrete cocompact subgroup, and $\iota(P) \leq T \times \mathfrak{t}^*$ is a discrete, cocompact, subgroup for a non-standard group*

structure on $(T \times \mathfrak{t}^*)/\iota(P)$ defined in terms of the Chern class (a certain bilinear form $c \colon \mathfrak{t}^* \times \mathfrak{t}^* \to \mathfrak{t}$), and the holonomy invariant $[\tau \colon P \to T]_{\exp(\mathcal{A})} \in \mathrm{Hom}_c(P, T)/\exp(\mathcal{A})$.

4) (M, ω) is equivariantly symplectomorphic to $\widetilde{\Sigma} \times_{\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)} T$ where the symplectic form and T-action are induced by the product ones, Σ is a good orbisurface, and the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ acts on $\widetilde{\Sigma} \times T$ diagonally, where the action of $\pi_1^{\mathrm{orb}}(\Sigma,\, p_0)$ on T is by means of any homomorphism $\mu \colon \pi_1^{\mathrm{orb}}(\Sigma) \to T$.

The proof of Theorem 5.1 uses as stepping stones the symplectic case treated in Section 4, and the coisotropic case treated in Section 3. Case 4) comes from Theorem 4.5 and Theorem 4.8; that is, the symplectic invariants and the construction of this case comes from these results. The fundamental observation to use the results in the aforementioned sections in order to prove Theorem 5.1 is that under the assumptions of Theorem 5.1, there are precisely two possibilities: a) either all of the T-orbits are symplectic 2-tori; or b) all of the 2-dimensional T-orbits are Lagrangian 2-tori. A significant part of the proof of Theorem 5.1 consists of unfolding item b) above into items 1), 2), 3) in the statement of Theorem 5.1. Notice that item 1) is classified in terms of the Delzant polytope, which is the only symplectic invariant in that case, see Remark 3.10, part (c).

6. Comments on the Proofs

In order to give an idea of the type of methods involved in the study of non Hamiltonian symplectic actions, we next give a outline of the proof of one of the results we have stated earlier: Theorem 3.7 (and readers can consult the references given througout the paper for the proofs of the other results, and for more detailed and general versions of them). For the notation and ingredients which we use next consult Section 3.3. We have divided the outline of the proof (in [7]) into three steps:

Step 1. *The orbit space.* First one shows that the orbit space M/T is a polyhedral $\mathfrak{l}^*$-parallel space, where $\mathfrak{l} := \ker(\omega^\mathfrak{t})$. Then one shows that, as $\mathfrak{l}^*$-parallel spaces, there is an isomorphism $M/T \simeq \Delta \times S$. Here Δ is a Delzant polytope, and S is a torus. The polytope Δ encodes the Hamiltonian part of the T-action, while the torus S encodes the free part of the action, see Section 3.6.2.

Step 2. *A nice connection.* We prove that there exists an admissible connection for the principal torus bundle $\pi \colon M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ (i.e. a "nice" linear assignment of lifts) $\xi \in \mathfrak{l}^* \mapsto L_\xi \in \mathcal{X}^\infty(M_{\mathrm{reg}})$, that has "simple" Lie brackets $[L_\xi, L_\eta]$ in the sense that $[L_\xi, L_\eta] = c(\xi, \eta)_M$ if $\xi, \eta \in N := (\mathfrak{l}/\mathfrak{t}_{\mathrm{h}})^*$ (c encodes Chern class of $\pi \colon M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$), and $[L_\xi, L_\eta] = 0$ otherwise, as well as "simple" symplectic pairings $\omega(L_\xi, L_\eta)$. The vector fields $L_\xi, \xi \in N$, have smooth extensions to M but are singular on $M \backslash M_{\mathrm{reg}}$ (see Section 3.6.2).

Step 3. *Distribution by symplectic toric manifolds.* We define the distribution $D_x := \mathrm{span}\{ L_\eta(x),\, Y_M(x) \mid Y \in \mathfrak{t}_{\mathrm{h}},\, \eta \in C \}$, $x \in M_{\mathrm{reg}}$, on M_{reg}

and prove that it is integrable, where $C \oplus N = \mathfrak{l}^*$. The integral manifolds to this distribution are T-equivariantly symplectomorphic to $(M_{\mathrm{h}}, \omega_{\mathrm{h}}, T_{\mathrm{h}})$. Next we give a natural construction of the groups H, G, and $G \times_H M_{\mathrm{h}}$ from the connection $\xi \mapsto L_\xi$. The group G involves the Chern class of $M_{\mathrm{reg}} \to M_{\mathrm{reg}}/T$ and H involves the holonomy of $\xi \mapsto L_\xi$. We gave the precise formulas in Section 3.4.3 and Section 3.4.4. Then we show that the following map is a T-equivariant symplectomorphism from $G \times_H M_{\mathrm{h}}$ to M: $((t, \xi), x) \mapsto t \cdot \mathrm{e}^{L_\xi}(x)$. This concludes the sketch of proof of Theorem 3.7.

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